MATHEMATICS

REDUCTION OF HYPERGEOMETRIC FUNCTIONS WITH INTEGRAL PARAMETER DIFFERENCES

BY

PER W. KARLSSON

(Communicated by Prof. C. J. Bouwkamp at the meeting of September 29, 1973)

Summary

A generalized hypergeometric function ${}_pF_q$ with negative integral differences between certain numerator and denominator parameters is proved to be in general expressible as a linear combination of lower-order functions. One possible extension of the result to hypergeometric functions of several variables is mentioned.

1. Generalized hypergeometric functions ${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z]$ with integral parameter differences are, broadly speaking, expressible as finite sums involving lower-order functions. Having previously obtained [1] a reduction formula for the case $a_1-b_1, \ldots, a_n-b_n \in \mathfrak{N}$ (a shorter proof of which was subsequently given by Srivastava [2]), we here consider the case $b_1-a_1, \ldots, b_n-a_n \in \mathfrak{N}$. It is supposed that (i) $n \leqq \min(p, q)$, (ii) $p \leqq q+1$, (iii) $|\arg(1-z)| < \pi$ when $p=q+1$, and (iv) no denominator parameter equals zero or a negative integer.

We first show that such a function is expressible as a linear combination of functions of the same type having parameter differences equal to unity:

$$(1)\quad \begin{cases} {}_pF_q\left[\begin{matrix} a_1, \ldots, & a_n, a_{n+1}, \ldots, a_p; \\ a_1+m_1+1, \ldots, a_n+m_n+1, & b_{n+1}, \ldots, b_q; \end{matrix} z\right] \prod\limits_{\mu=1}^{n} \dfrac{m_\mu!}{(a_\mu, m_\mu+1)} = \\ \qquad = \sum\limits_{j_1=0}^{m_1} \cdots \sum\limits_{j_n=0}^{m_n} \prod\limits_{\mu=1}^{n} \dfrac{(-m_\mu, j_\mu)}{j_\mu!(a_\mu+j_\mu)} \times \\ \qquad\qquad \times\, {}_pF_q\left[\begin{matrix} a_1+j_1, \ldots, & a_n+j_n, a_{n+1}, \ldots, a_p; \\ a_1+j_1+1, \ldots, a_n+j_n+1, & b_{n+1}, \ldots, b_q; \end{matrix} z\right] \end{cases}$$

Here, $m_1, \ldots, m_n$ are non-negative integers, and

$$(2)\qquad (c, s) \equiv \Gamma(c+s)/\Gamma(c)$$

denotes the Pochhammer symbol. To prove (1), we replace the hypergeometric functions by power series. (For $p=q+1$ the proof thus requires that $|z|<1$; the result, however, follows for $|\arg(1-z)|<\pi$ by analytic continuation.) The right-hand member R then reads

$$R=\sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n}\prod_{\mu=1}^{n}\frac{(-m_\mu,j_\mu)}{j_\mu!\,(a_\mu+j_\mu)}\sum_{k=0}^{\infty}B\prod_{\mu=1}^{n}\frac{(a_\mu+j_\mu,k)}{(a_\mu+j_\mu+1,k)},$$

where, for brevity,

$$B\equiv\frac{(a_{n+1},k)\ldots(a_p,k)z^k}{(b_{n+1},k)\ldots(b_q,k)k!}; \tag{3}$$

empty products are interpreted as unity. Now, by the aid of the elementary rules for the Pochhammer symbol, and of Vandermonde's theorem

$${}_2F_1[-m,\beta;\gamma;1]=(\gamma-\beta,m)/(\gamma,m), \tag{4}$$

R is further transformed as follows:

$$\begin{aligned}R&=\sum_{k=0}^{\infty}\sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n}B\prod_{\mu=1}^{n}\left\{\frac{(-m_\mu,j_\mu)}{j_\mu!\,(a_\mu+j_\mu)}\frac{a_\mu+j_\mu}{a_\mu+j_\mu+k}\right\}\\
&=\sum_{k=0}^{\infty}B\prod_{\mu=1}^{n}\sum_{j_\mu=0}^{m_\mu}\frac{(-m_\mu,j_\mu)\,(a_\mu+k,j_\mu)}{j_\mu!\,(a_\mu+k)\,(a_\mu+k+1,j_\mu)}\\
&=\sum_{k=0}^{\infty}B\prod_{\mu=1}^{n}\frac{{}_2F_1[-m_\mu,a_\mu+k;a_\mu+k+1;1]}{a_\mu+k}\\
&=\sum_{k=0}^{\infty}B\prod_{\mu=1}^{n}\frac{(1,m_\mu)}{(a_\mu+k)\,(a_\mu+k+1,m_\mu)}\\
&=\sum_{k=0}^{\infty}B\prod_{\mu=1}^{n}\frac{m_\mu!\,(a_\mu,k)}{(a_\mu,k+1+m_\mu)}\\
&=\sum_{k=0}^{\infty}B\prod_{\mu=1}^{n}\frac{m_\mu!\,(a_\mu,k)}{(a_\mu,m_\mu+1)\,(a_\mu+m_\mu+1,k)}.\end{aligned}$$

This expression is, upon insertion of (3), easily seen to be in fact the left-hand member of (1).

For functions of the type appearing on the right-hand side of (1) we next show that, if the parameters $a_1,\ldots,a_n$ are all different, then

$$\left\{\begin{aligned}&{}_pF_q\left[\begin{matrix}a_1,\ldots, & a_n,a_{n+1},\ldots,a_p;\\ a_1+1,\ldots, & a_n+1,b_{n+1},\ldots,b_q;\end{matrix}\,z\right]=\\
&\qquad=\sum_{i=1}^{n}{}_{p-n+1}F_{q-n+1}\left[\begin{matrix}a_i,a_{n+1},\ldots,a_p;\\ a_i+1,b_{n+1},\ldots,b_q;\end{matrix}\,z\right]\prod_{\substack{\mu=1\\ \mu\neq i}}^{n}\frac{a_\mu}{a_\mu-a_i}.\end{aligned}\right. \tag{5}$$

To prove this result, the left-hand member is rewritten:

$$\sum_{k=0}^{\infty} B \prod_{\mu=1}^{n} \frac{(a_\mu, k)}{(a_\mu+1, k)} = \prod_{\mu=1}^{n} a_\mu \sum_{k=0}^{\infty} B \prod_{\mu=1}^{n} (a_\mu+k)^{-1} =$$

$$= \prod_{\mu=1}^{n} a_\mu \sum_{k=0}^{\infty} B \sum_{i=1}^{n} [(a_i+k) \prod_{\substack{\mu=1\\ \mu\neq i}}^{n} (a_\mu-a_i)]^{-1} =$$

$$= \sum_{i=1}^{n} \left\{ \prod_{\substack{\mu=1\\ \mu\neq i}}^{n} \frac{a_\mu}{a_\mu-a_i} \right\} \sum_{k=0}^{\infty} \frac{(a_i, k)B}{(a_i+1, k)},$$

which is, obviously, the right-hand side; the second equals sign does require that all elements of $\{a_1, \ldots, a_n\}$ be different.

Finally, combining (1) and (5) we find the reduction formula

$$\text{(6)}\quad \left\{ \begin{aligned} &{}_pF_q \left[\begin{matrix} a_1, \ldots, & a_n, a_{n+1}, \ldots, a_p; \\ a_1+m_1+1, \ldots, & a_n+m_n+1, b_{n+1}, \ldots, b_q; \end{matrix} z \right] \prod_{\mu=1}^{n} \frac{m_\mu!}{(a_\mu, m_\mu+1)} = \\ &= \sum_{i=1}^{n} \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} C \;\; {}_{p-n+1}F_{q-n+1} \left[\begin{matrix} a_i+j_i, a_{n+1}, \ldots, a_p; \\ a_i+j_i+1, b_{n+1}, \ldots, b_q; \end{matrix} z \right], \end{aligned} \right.$$

where

$$\text{(7)}\qquad C = \frac{1}{a_i+j_i} \prod_{\nu=1}^{n} \frac{(-m_\nu, j_\nu)}{j_\nu!} \prod_{\substack{\mu=1\\ \mu\neq 1}}^{n} \frac{1}{a_\mu+j_\mu-a_i-j_i},$$

provided that (i) all elements of $\{a_1, \ldots, a_n\}$ are different and (ii) if $a_\mu - a_\nu = N \in \mathfrak{N}$, then $N > m_\nu$ is satisfied. It should be noted that, for $n=1$, formulae (1) and (6) are identical, and reduction to lower-order functions does not take place.

A simple example of reduction is obtained by taking $p=q=n=2$, $m_1=m_2=0$; equation (6) then yields

$$\text{(8)}\quad \left\{ \begin{aligned} &{}_2F_2 \left[\begin{matrix} a_1, a_2; \\ a_1+1, a_2+1; \end{matrix} z \right] = \frac{a_2}{a_2-a_1} {}_1F_1 \left[\begin{matrix} a_1; \\ a_1+1; \end{matrix} z \right] + \\ &+ \frac{a_1}{a_1-a_2} {}_1F_1 \left[\begin{matrix} a_2; \\ a_2+1; \end{matrix} z \right], \end{aligned} \right.$$

provided that $a_2 \neq a_1$; this ${}_2F_2$ is thus expressible in terms of incomplete gamma functions.

2. The rather passive role of the factor B in the preceding transformations suggests the possibility of establishing similar results involving hypergeometric functions of several variables. We briefly mention one example of this kind. (A general formula would be extremely complicated.)

One of the generalized Kampé de Fériet functions is defined by

$$
(9)\quad \left\{
\begin{aligned}
&F\,{}^{n:1}_{n:0}\left[\begin{matrix} a_1, \ldots, a_n : b_1, \ldots, b_r; \\ c_1, \ldots, c_n : \qquad\quad ; \end{matrix} z_1, \ldots, z_r\right] = \\
&\qquad = \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \prod_{\mu=1}^{n} \frac{(a_\mu, k_1+\ldots+k_r)}{(c_\mu, k_1+\ldots+k_r)} \prod_{\nu=1}^{r} \frac{(b_\nu, k_\nu)\, z_\nu^{k_\nu}}{k_\nu!}\,;
\end{aligned}\right.
$$

it reduces to Lauricella's F_D of r variables when $n=1$. A reduction formula similar to (6) is

$$
(10)\quad \left\{
\begin{aligned}
&F\,{}^{n:1}_{n:0}\left[\begin{matrix} a_1, \ldots, \qquad a_n : b_1, \ldots, b_r; \\ a_1+m_1+1, \ldots, a_n+m_n+1 : \qquad ; \end{matrix} z_1, \ldots, z_r\right] \times \\
&\qquad \times \prod_{\mu=1}^{n} \frac{m_\mu!}{(a_\mu, m_\mu+1)} = \\
&= \sum_{i=1}^{n} \sum_{j^*=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} C \quad F_D\left[\begin{matrix} a_i+j_i : b_1, \ldots, b_r; \\ a_i+j_i+1 : \qquad ; \end{matrix} z_1, \ldots, z_r\right],
\end{aligned}\right.
$$

where C is the quantity defined in equation (7). Again, if a parameter difference $a_\mu - a_\nu$ equals a non-negative integer N, then the condition $N > m_\nu$ must be satisfied. The proof of (10) is quite similar to that of (6).

Department of Physics,
The Technical University,
Lyngby, Denmark

REFERENCES

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